

# Exponents of inhomogeneous Diophantine Approximation

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**Abstract** – In Diophantine Approximation, inhomogeneous problems are linked with homogeneous ones by means of the so-called Transference Theorems. We revisit this classical topic by introducing new exponents of Diophantine approximation. We prove that the exponent of approximation to a generic point in  $\mathbf{R}^n$  by a system of  $n$  linear forms is equal to the inverse of the uniform homogeneous exponent associated to the system of dual linear forms.

## 1. Introduction and results.

It is a well known fact that inhomogeneous problems in Diophantine Approximation are connected to homogeneous ones by means of the so-called Transference Theorems. We revisit this classical topic, referring mainly to the book of Cassels [6], in the light of recent results on uniform exponents of Diophantine approximation, which have been introduced in a restricted setting in [5]. These uniform exponents, indicated by a ‘hat’, enable us here to control quantitatively the accuracy of the inhomogeneous approximation at a generic point in the ambient space.

Let us begin with some notations and definitions. If  $\underline{\theta}$  is a (column) vector in  $\mathbf{R}^n$ , we denote by  $|\underline{\theta}|$  the maximum of the absolute values of its coordinates and by

$$\|\underline{\theta}\| = \min_{\underline{x} \in \mathbf{Z}^n} |\underline{\theta} - \underline{x}|$$

the maximum of the distances of its coordinates to the rational integers.

Let  $n$  and  $m$  be two positive integers and let  $A$  be a real matrix with  $n$  rows and  $m$  columns. For any  $n$ -tuple  $\underline{\theta}$  of real numbers, we denote by  $w(A, \underline{\theta})$  the supremum of the real numbers  $w$  for which, *for arbitrarily large real numbers  $X$* , the inequalities

$$(1) \quad \|A\underline{x} + \underline{\theta}\| \leq X^{-w} \quad \text{and} \quad |\underline{x}| \leq X$$

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have a solution  $\underline{x}$  in  $\mathbf{Z}^m$ . Following the conventions of [5], we denote by  $\hat{w}(A, \underline{\theta})$  the supremum of the real numbers  $w$  for which, *for all sufficiently large positive real numbers*  $X$ , the inequalities (1) have an integer solution  $\underline{x}$  in  $\mathbf{Z}^m$ . The lower bounds

$$w(A, \underline{\theta}) \geq \hat{w}(A, \underline{\theta}) \geq 0$$

are then obvious. We define furthermore two homogeneous exponents  $w(A)$  and  $\hat{w}(A)$  as in (1) with  $\underline{\theta} = {}^t(0, \dots, 0)$ , requiring moreover that the integer solution  $\underline{x}$  should be non-zero. Then the Dirichlet box principle implies that

$$w(A) \geq \hat{w}(A) \geq \frac{m}{n}.$$

In this respect, Jarnik [14] has given non trivial lower bounds of  $w(A)$  in terms of  $\hat{w}(A)$ . Furthermore, it follows from the Borel–Cantelli Lemma that the equalities

$$w(A) = \hat{w}(A) = \frac{m}{n}$$

hold for almost all matrices  $A$  in  $M_{n,m}(\mathbf{R})$  with respect to the Lebesgue measure on  $\mathbf{R}^{mn}$  (see also Groshev [12] ; his result is sharper, actually).

The transposed matrix of any matrix  $A$  is denoted by  ${}^tA$ . We can now state our main result.

**Theorem.** *For any  $n$ -tuple  $\underline{\theta}$  of real numbers, we have the lower bounds*

$$(2) \quad w(A, \underline{\theta}) \geq \frac{1}{\hat{w}({}^tA)} \quad \text{and} \quad \hat{w}(A, \underline{\theta}) \geq \frac{1}{w({}^tA)},$$

*with equality in (2) for almost all  $\underline{\theta}$  with respect to the Lebesgue measure on  $\mathbf{R}^n$ .*

Let us first examine the simplest case of a  $1 \times 1$  matrix  $A = (\xi)$ , where  $\xi$  is an irrational real number. The Dirichlet box principle asserts that for any real number  $Q \geq 1$ , there exists an integer  $q$  such that

$$1 \leq |q| \leq Q \quad \text{and} \quad \|q\xi\| \leq Q^{-1}.$$

This statement is best possible in the sense that the exponent  $-1$  in the upper bound  $Q^{-1}$  cannot be replaced by a smaller value (see e.g. [15], [9] or [22], page 26). It then follows that our exponent  $\hat{w}({}^tA)$  is equal to 1, as well as the generic inhomogeneous exponent  $\inf_{\theta \in \mathbf{R}} w(A, \theta)$ . In fact, a more precise result holds in this particular case. Namely, Minkowski has proved that for any real number  $\theta$ , the system of inequalities

$$|q| \leq Q \quad \text{and} \quad \|q\xi + \theta\| \leq \frac{1}{4}Q^{-1}$$

has an integer solution  $q$  for infinitely many integers  $Q$ . Moreover, our Theorem shows that  $-1$  is the best possible exponent, regardless of the irrational number  $\xi$ . Besides, Cassels has observed that there does not exist any inhomogeneous analogue of the Dirichlet box principle, even if we weaken the property of approximation. In Theorem III of Chapter 3 from [6], he constructed a Liouville number  $\xi$  and a real number  $\theta$  such that, for any  $\epsilon > 0$ , we have the lower bound

$$\min_{|q| \leq Q} \|q\xi + \theta\| \geq Q^{-\epsilon}$$

for infinitely many integers  $Q$ . The result of Cassels follows immediately from our Theorem, since  $w((\xi)) = +\infty$  for any Liouville number  $\xi$ . The uniform exponent of inhomogeneous approximation  $\hat{w}((\xi), \theta)$  therefore vanishes for almost all  $\theta$ .

The cases  $(m, n) = (1, 2)$  and  $(m, n) = (2, 1)$  have been studied by Khintchine [15]. Notice that the metrical statement of our Theorem is of same spirit as Satz V b from [15].

We could refine the first statement of the Theorem (and of Proposition 1 below) by taking into account whether or not there exists a positive constant  $c$  such that, *for arbitrarily large real numbers  $X$* , the inequalities

$$\|A\underline{x} + \underline{\theta}\| \leq c X^{-w(A, \underline{\theta})} \quad \text{and} \quad |\underline{x}| \leq X$$

have a solution  $\underline{x}$  in  $\mathbf{Z}^m$ . We take into consideration this remark in the statement of the Corollary below.

The next result should be compared with Theorem X of Chapter 5 from [6].

**Proposition 1.** *Let  $A$  be a matrix in  $M_{n,m}(\mathbf{R})$ . For any exponent  $w > 1/\hat{w}({}^tA)$ , there exists a real  $n$ -tuple  $\underline{\theta}$  such that the lower bound*

$$\|A\underline{x} + \underline{\theta}\| \geq |\underline{x}|^{-w}$$

*holds for any integer  $m$ -tuple  $\underline{x}$  whose norm  $|\underline{x}|$  is sufficiently large. Moreover, there exists some real  $n$ -tuple  $\underline{\theta}$  such that*

$$\|A\underline{x} + \underline{\theta}\| \geq \frac{1}{72n^2(8m)^{m/n}} |\underline{x}|^{-m/n}$$

*holds for all non-zero integer  $m$ -tuples  $\underline{x}$ .*

Cassels [6], page 85, has proved the second assertion of Proposition 1, without however computing the constant occurring in the right-handside of the lower bound. Moreover, our first assertion improves Theorem X of [6] whenever  $\hat{w}({}^tA) > n/m$ . See Section 5 for examples of such matrices  $A$ .

If the subgroup  $G = {}^tAZ^n + \mathbf{Z}^m$  of  $\mathbf{R}^m$  generated by the  $n$  rows of  $A$  together with  $\mathbf{Z}^m$  has maximal rank  $m + n$ , then Kronecker's Theorem asserts that the dual subgroup  $\Gamma = AZ^m + \mathbf{Z}^n$  of  $\mathbf{R}^n$  generated by the  $m$  columns of  $A$  and by  $\mathbf{Z}^n$  is dense in  $\mathbf{R}^n$ . With this respect, our Theorem may be viewed as a measure of the density of  $\Gamma$ . In the case where the rank of  $G$  is  $< m + n$ , we clearly have

$$\hat{w}({}^tA) = w({}^tA) = +\infty \quad \text{and} \quad \hat{w}(A, \underline{\theta}) = w(A, \underline{\theta}) = 0$$

for any  $n$ -tuples  $\underline{\theta}$  located outside a discrete family of parallel hyperplanes in  $\mathbf{R}^n$ . The assertion of the Theorem is then obvious. In the sequel of the paper, we shall therefore assume that the rank over  $\mathbf{Z}$  of the group  $G$  is equal to  $m + n$ . Notice however that the exponent  $\hat{w}({}^tA)$  may be infinite, even when  $G$  has rank  $m + n$ , as proved by Khintchine [15] in the case  $(m, n) = (1, 2)$  and by Jarník [13] in the general case  $(m, n)$  with  $n \geq 2$  (when  $n = 1$  and  $m$  is arbitrary, the exponent  $\hat{w}({}^tA)$  can be as large as 1 [13], but not larger [15]). See also Theorem XIV (page 94) of [6] for the construction of such a matrix  $A$  and the following Theorem XV concerning the density of the associated group  $\Gamma$ .

Let us illustrate our Theorem by the example of the row (resp. column) matrices

$$A = (\xi, \dots, \xi^n), \quad \text{resp.} \quad A = {}^t(\xi, \dots, \xi^n),$$

made up with the successive powers of a transcendental real number  $\xi$ . Then, the corresponding exponents  $\hat{w}(A)$  are uniformly bounded in terms of  $n$  (see [5] for references). Roy [19] determined these exponents for  $n = 2$  when  $\xi$  is a Fibonacci continued fraction, that is, when we have

$$\xi = [0; a, b, a, a, b, a, b, \dots],$$

where the sequence of partial quotients of  $\xi$  is given by the fixed point of the Fibonacci substitution  $a \rightarrow ab, b \rightarrow a$ . Here,  $a$  and  $b$  denote distinct positive integers.

Combining these results with our Theorem, we obtain the following statement.

**Corollary.** *Let  $n$  be a positive integer and let  $\xi$  be a real transcendental number.*

(i) *There exists a positive constant  $c$  such that, for any real number  $\theta$ , there exist infinitely many polynomials  $P(X)$  with integer coefficients, degree at most  $n$ , and*

$$|P(\xi) + \theta| \leq cH(P)^{-\lceil n/2 \rceil}.$$

(ii) *There exists a positive constant  $c$  such that, for any real  $n$ -tuple  $\underline{\theta} = (\theta_1, \dots, \theta_n)$ , there exist infinitely many integers  $q$  with*

$$\max_{1 \leq j \leq n} \|q\xi^j + \theta_j\| \leq c|q|^{-1/(2n-1)}.$$

(iii) When  $n = 2$ , the assertions (i) and (ii) remain valid with the exponents

$$(1 + \sqrt{5})/2 \simeq 1.618... \quad \text{and} \quad (3 - \sqrt{5})/2 \simeq 0.3819...,$$

respectively. If moreover  $\xi$  is a Fibonacci continued fraction and if

$$w > (1 + \sqrt{5})/2 \quad \text{and} \quad \lambda > (3 - \sqrt{5})/2$$

then, for almost all real numbers  $\theta$  and for almost all pairs of real numbers  $(\theta_1, \theta_2)$ , we have the respective lower bounds

$$|P(\xi) + \theta| \geq H(P)^{-w} \quad \text{and} \quad \max\{\|q\xi + \theta_1\|, \|q\xi^2 + \theta_2\|\} \geq |q|^{-\lambda},$$

for any polynomial  $P(X)$  with integer coefficients and degree  $\leq 2$  and with sufficiently large height, and for any integer  $q$  with sufficiently large absolute value.

Thus, for a Fibonacci continued fraction  $\xi$ , the critical exponents in degree  $n = 2$  for the problems of inhomogeneous approximation (i) and (ii) are respectively  $w = 1.618...$  and  $\lambda = 0.3819...$ , instead of the exponents 2 and  $1/2$ , which occur in the generic situation. Notice that  $\hat{w}(A)$  and  $\hat{w}({}^tA)$  have also been determined for  $A = (\xi, \xi^2)$  when  $\xi$  is a Sturmian continued fraction, see [5]. This provides further examples of matrices  $A$  with  $\hat{w}(A)$  and  $\hat{w}({}^tA)$  greater than in the generic case.

To conclude this introduction, let us recall the inequalities relying the exponents  $w(A)$  and  $w({}^tA)$ , which follow from the Khintchine transference principle (cf. for example [22], Theorem 5C, page 99) ; with the preceding notations, we have

$$w(A) \geq \frac{m w({}^tA) + m - 1}{(n - 1)w({}^tA) + n}.$$

Furthermore, a careful reading of the proof shows that the uniform exponents  $\hat{w}(A)$  and  $\hat{w}({}^tA)$  are linked by the same relation

$$\hat{w}(A) \geq \frac{m \hat{w}({}^tA) + m - 1}{(n - 1)\hat{w}({}^tA) + n}.$$

Our article is organized as follows. Section 2 is devoted to the definition and to the properties of the sequence of best approximations. A crucial fact is that it increases at least geometrically. Some transference lemma is stated and proved in Section 3. It is used in the next Section, where we establish the Theorem and Proposition 1. The Corollary is then discussed in Section 5. Finally, questions of Hausdorff dimensions, which arise naturally from the Theorem, are briefly treated in Section 6.

## 2. Best approximations.

Following the notations of [6], we denote by

$$M_j(\underline{y}) = \sum_{i=1}^n \alpha_{i,j} y_i, \quad \underline{y} = {}^t(y_1, \dots, y_n), \quad (1 \leq j \leq m)$$

the linear forms determined by the columns of the matrix  $A = (\alpha_{i,j})$  and we set

$$M(\underline{y}) = \max_{1 \leq j \leq m} \|M_j(\underline{y})\| = \|{}^t A \underline{y}\|.$$

Observe that the quantity  $M(\underline{y})$  is positive for all non-zero integer  $n$ -tuples  $\underline{y}$ , since we have assumed that the rank over  $\mathbf{Z}$  of the group  $G$  is equal to  $m+n$ . Then, we can build inductively a sequence of integer vectors

$$\underline{y}_i = {}^t(y_{i,1}, \dots, y_{i,n}), \quad (i \geq 1),$$

called *a sequence of best approximations* (\*) related to the linear forms  $M_1, \dots, M_m$  and to the supremum norm, which satisfies the following properties. Set

$$|\underline{y}_i| = Y_i \quad \text{and} \quad M_i = M(\underline{y}_i).$$

Then, we have

$$1 = Y_1 < Y_2 < \dots \quad \text{and} \quad M_1 > M_2 > \dots,$$

and  $M(\underline{y}) \geq M_i$  for all non-zero integer vectors  $\underline{y}$  of norm  $|\underline{y}| < Y_{i+1}$ . We start the construction with a smallest *minimal point*  $\underline{y}_1$  in the sense of [8], verifying  $Y_1 = |\underline{y}_1| = 1$  and  $M(\underline{y}) \geq M(\underline{y}_1) = M_1$  for any integer point  $\underline{y} \in \mathbf{Z}^n$  with norm  $|\underline{y}| = 1$ . Suppose that  $\underline{y}_1, \dots, \underline{y}_i$  have already been constructed in such a way that  $M(\underline{y}) \geq M_i$  for all non-zero integer point  $\underline{y}$  of norm  $|\underline{y}| \leq Y_i$ . Let  $Y$  be the smallest positive integer  $> Y_i$  for which there exists an integer point  $\underline{z}$  verifying  $|\underline{z}| = Y$  and  $M(\underline{z}) < M_i$ . The integer  $Y$  does exist by the Dirichlet box principle since  $M_i > 0$ . Among those points  $\underline{z}$ , we select an element  $\underline{y}$  for which  $M(\underline{z})$  is minimal. We then set

$$\underline{y}_{i+1} = \underline{y}, \quad Y_{i+1} = Y \quad \text{and} \quad M_{i+1} = M(\underline{y}).$$

The sequence  $(\underline{y}_i)_{i \geq 1}$  thus obtained satisfies clearly the desired properties.

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(\*) According to [17], a best approximation should be a vector belonging to  $\mathbf{Z}^{n+m}$ , by analogy with the usual continued fraction process. We forget here the last  $m$  coordinates which are insignificant for our purpose.

Let  $w$  be a real number  $< \hat{w}({}^tA)$ , so that the system of inequalities

$$M(\underline{y}) \leq Y^{-w} \quad \text{and} \quad |\underline{y}| \leq Y$$

has a non-zero integer solution  $\underline{y}$  for any sufficiently large  $Y$ . Choosing  $Y < Y_{i+1}$  arbitrarily close to  $Y_{i+1}$ , we obtain the upper bound

$$(3) \quad M_i \leq Y_{i+1}^{-w}$$

for any sufficiently large index  $i$ , using the characteristic property of the best approximations.

Suppose now that  $w < w({}^tA)$ . Then, there exist infinitely many indices  $i$  for which

$$(4) \quad M_i \leq Y_i^{-w}.$$

The indices  $i$  satisfying (4) are obtained by inserting the norm

$$Y_i \leq |\underline{y}| < Y_{i+1}$$

of the integer solutions  $\underline{y}$  of the inequation  $M(\underline{y}) \leq |\underline{y}|^{-w}$  in the sequence  $(Y_k)_{k \geq 1}$ .

Observe furthermore that the Dirichlet box principle (cf. [6], Theorem VI, page 13) ensures that the system of inequations

$$M(\underline{y}) \leq Y^{-n/m} \quad \text{and} \quad |\underline{y}| \leq Y$$

has a non-zero integer solution  $\underline{y}$  for any  $Y \geq 1$ . Arguing as above, we obtain the upper bound

$$(5) \quad M_i \leq Y_{i+1}^{-n/m}$$

for all  $i \geq 1$ .

**Lemma 1.** *There exists a positive constant  $c$  such that*

$$Y_i \geq c 2^{i/(3^{m+n}-1)}$$

for all  $i \geq 1$ .

*Proof.* Lagarias [17] has established in a quite general framework that a sequence of best approximations increases at least geometrically. We take again the argumentation used in the proof of Theorem 2.2 from [16]. Let us consider the  $3^{m+n} + 1$  consecutive vectors

$$\underline{y}_i, \underline{y}_{i+1}, \dots, \underline{y}_{i+3^{m+n}}.$$

By the usual box principle, there exist two indices  $r$  and  $s$ , with  $0 \leq r < s \leq 3^{m+n}$ , such that

$$\underline{y}_{i+r,j} \equiv \underline{y}_{i+s,j} \pmod{3} \quad \text{for all } j = 1, \dots, n,$$

and

$$\langle M_k(\underline{y}_{i+r}) \rangle \equiv \langle M_k(\underline{y}_{i+s}) \rangle \pmod{3} \quad \text{for all } k = 1, \dots, m,$$

where the notation  $\langle x \rangle$  stands for the closest integer to the real number  $x$ . Setting

$$\underline{z} = \frac{\underline{y}_{i+s} - \underline{y}_{i+r}}{3},$$

we have

$$|\underline{z}| \leq \frac{Y_{i+s} + Y_{i+r}}{3} \quad \text{and} \quad M(\underline{z}) \leq \frac{M_{i+r} + M_{i+s}}{3} < M_{i+r}.$$

Since  $\underline{z}$  is a non-zero integer vector, we get

$$Y_{i+r+1} \leq \frac{Y_{i+s} + Y_{i+r}}{3}$$

and

$$Y_{i+3^{m+n}} \geq Y_{i+s} \geq 3Y_{i+r+1} - Y_{i+r} \geq 2Y_{i+r+1} \geq 2Y_{i+1}$$

for any  $i \geq 1$ . The expected lower bound then follows by induction on  $i$ .  $\square$

**Lemma 2.** *For almost all real  $n$ -tuples  $\underline{\theta} = {}^t(\theta_1, \dots, \theta_n)$ , we have the lower bound*

$$\|y_{i,1}\theta_1 + \dots + y_{i,n}\theta_n\| \geq Y_i^{-\delta},$$

for any  $\delta > 0$ , and any index  $i$  which is sufficiently large in terms of  $\delta$  and of  $\theta_1, \dots, \theta_n$ .

*Proof.* We can assume without restriction that the numbers  $\theta_j$  are located in the interval  $[0, 1]$  and that  $\delta$  is given. For a fixed  $i$ , the reverse inequality

$$(6) \quad \|y_{i,1}\theta_1 + \dots + y_{i,n}\theta_n\| < Y_i^{-\delta}$$

defines in the hypercube  $[0, 1]^n$  a subset of Euclidean volume  $\leq 2\sqrt{n}Y_i^{-\delta}$ . By Lemma 1, the series  $\sum_{i \geq 1} Y_i^{-\delta}$  converges. It then follows from the Borel–Cantelli Lemma that the set of  $\underline{\theta}$  satisfying (6) for infinitely many indices  $i$  has Lebesgue measure zero.  $\square$

### 3. A transference lemma.

Let us consider now the linear forms

$$L_i(\underline{x}) = \sum_{j=1}^m \alpha_{i,j} x_j, \quad \underline{x} = {}^t(x_1, \dots, x_m), \quad (1 \leq i \leq n),$$

determined by the rows of the matrix  $A$ . The following result relies the problem of the inhomogeneous simultaneous approximation by the linear forms  $L_i$  to the problem of homogeneous simultaneous approximation by the linear forms  $M_j$ .

**Lemma 3.** Set  $\kappa = 2^{1-m-n}((m+n)!)^2$ . Let  $X$  and  $Y$  be two positive real numbers. Suppose that we have the lower bound

$$M(\underline{y}) \geq \kappa X^{-1}$$

for any non-zero integer  $n$ -tuple  $\underline{y}$  of norm  $|\underline{y}| \leq Y$ . Then, for all real  $n$ -tuples  $(\theta_1, \dots, \theta_n)$ , there exists an integer  $m$ -tuple  $\underline{x}$  with norm  $|\underline{x}| \leq X$  such that

$$\max_{1 \leq i \leq n} \|L_i(\underline{x}) + \theta_i\| \leq \kappa Y^{-1}.$$

*Proof.* This is the first assertion of Lemma 4.1 from [23]. For the convenience of the reader, we reproduce the proof. Let  $X$  and  $Z$  be two positive real numbers. Part B of Theorem XVII in chapter V of [6] asserts that the system of inequations

$$\max_{1 \leq i \leq n} \|L_i(\underline{x}) + \theta_i\| \leq Z \quad \text{and} \quad |\underline{x}| \leq X$$

has an integer solution  $\underline{x} \in \mathbf{Z}^m$ , provided that the upper bound

$$(7) \quad \|y_1\theta_1 + \dots + y_n\theta_n\| \leq \kappa^{-1} \max\{XM(\underline{y}), Z|\underline{y}|\}$$

holds for all integer  $n$ -tuples  $\underline{y}$ . Let us apply this result with  $Z = \kappa Y^{-1}$ . The condition (7) is satisfied when  $|\underline{y}| \geq Y$ , since then  $\kappa^{-1}Z|\underline{y}|$  is  $\geq 1$  while the left hand side of (7) is  $\leq 1/2$ . If  $\underline{y}$  is non-zero and  $|\underline{y}| \leq Y$ , our assumption ensures that  $M(\underline{y}) \geq \kappa X^{-1}$ , and the right hand side of (7) is  $\geq 1$  in this case too. Finally, (7) obviously holds for  $\underline{y} = 0$ .  $\square$

Up to the value of the numerical constant  $\kappa$ , the above mentioned Theorem XVII of [6] is a consequence of the more general Theorem VI in Chapter XI of [7], when applied to the distance function

$$F(x_1, \dots, x_{m+n}) = X^{-1} \left( \sum_{j=1}^m |x_j| \right) + Z^{-1} \left( \sum_{i=1}^n |L_i(x_1, \dots, x_m) + x_{m+i}| \right)$$

in  $\mathbf{R}^{m+n}$ . Notice that this last result provides also an explicit construction of the approximating point  $\underline{x}$  in terms of successive minima and of duality.

#### 4. Proof of the Theorem and of Proposition 1.

First, we prove that the lower bounds

$$(8) \quad w(A, \underline{\theta}) \geq \frac{1}{\hat{w}({}^t A)} \quad \text{and} \quad \hat{w}(A, \underline{\theta}) \geq \frac{1}{w({}^t A)}$$

hold for all real  $n$ -tuples  $\underline{\theta} = {}^t(\theta_1, \dots, \theta_n)$ .

Let  $w > \hat{w}({}^t A)$  be a real number. By definition of the exponent  $\hat{w}({}^t A)$ , there exists a real number  $Y$ , which may be chosen arbitrarily large, such that

$$(9) \quad M(\underline{y}) \geq Y^{-w}$$

for any non-zero integer  $n$ -tuple  $\underline{y}$  of norm  $|\underline{y}| \leq Y$ . We use Lemma 3 with  $X = \kappa Y^w$ , where  $\kappa = 2^{1-m-n}((m+n)!)^2$ . Consequently, there exists an integer  $m$ -tuple  $\underline{x}$  of norm  $|\underline{x}| \leq X$  such that

$$\max_{1 \leq i \leq n} \|L_i(\underline{x}) + \theta_i\| \leq \kappa Y^{-1} = \kappa^{(1+1/w)} X^{-1/w} \leq \kappa^{(1+1/w)} |\underline{x}|^{-1/w}.$$

We deduce that  $w(A, \underline{\theta}) \geq 1/w$ . The first assertion of (8) then follows by letting  $w$  tend to  $\hat{w}({}^t A)$ .

The second lower bound of (8) is established along the same lines, observing that for  $w > w({}^t A)$  and any sufficiently large real number  $Y$ , inequality (9) is satisfied for any non-zero integer  $n$ -tuple  $\underline{y}$  of norm  $|\underline{y}| \leq Y$ .

We shall now prove that the inverse upper bounds

$$(10) \quad w(A, \underline{\theta}) \leq \frac{1}{\hat{w}({}^t A)} \quad \text{and} \quad \hat{w}(A, \underline{\theta}) \leq \frac{1}{w({}^t A)}$$

hold for almost all real  $n$ -tuples  $\underline{\theta} = {}^t(\theta_1, \dots, \theta_n)$ .

The duality formula  ${}^t \underline{y} A \underline{x} = {}^t \underline{x} {}^t A \underline{y}$  written in the form

$$y_1 \theta_1 + \dots + y_n \theta_n = \sum_{i=1}^n y_i (L_i(x_1, \dots, x_m) + \theta_i) - \sum_{j=1}^m x_j M_j(y_1, \dots, y_n)$$

implies the upper bound

$$(11) \quad \|y_1 \theta_1 + \dots + y_n \theta_n\| \leq n |\underline{y}| \max_{1 \leq i \leq n} \|L_i(\underline{x}) + \theta_i\| + m |\underline{x}| M(\underline{y})$$

for all integer vectors  $\underline{x} = {}^t(x_1, \dots, x_m)$  and  $\underline{y} = {}^t(y_1, \dots, y_n)$ .

Let

$$\underline{y}_i = {}^t(y_{i,1}, \dots, y_{i,n}) \quad \text{and} \quad Y_i = |\underline{y}_i| \quad (i \geq 1)$$

be a sequence of best approximations relative to the matrix  ${}^tA$ . Suppose that for all  $\delta > 0$  we have the lower bound

$$(12) \quad \|y_{i,1}\theta_1 + \cdots + y_{i,n}\theta_n\| \geq Y_i^{-\delta}$$

for any index  $i$  large enough. By Lemma 2, the estimation (12) holds for almost all real  $n$ -tuple  $\underline{\theta}$ . Let us fix now two real numbers  $\delta$  and  $w$  such that

$$0 < \delta < w < \hat{w}({}^tA).$$

Let  $\underline{x}$  be an integer  $m$ -tuple with sufficiently large norm  $|\underline{x}|$ , and let  $k$  be the index defined by the inequalities

$$Y_k \leq (2m|\underline{x}|)^{1/(w-\delta)} < Y_{k+1},$$

so that

$$Y_{k+1}^w > (2m|\underline{x}|)^{w/(w-\delta)} \geq 2m|\underline{x}|Y_k^\delta.$$

Combining now the estimations (3), (11) with  $\underline{y} = \underline{y}_k$  and (12) for  $i = k$ , we obtain

$$\begin{aligned} Y_k^{-\delta} &\leq n|\underline{y}_k| \max_{1 \leq i \leq n} \|L_i(\underline{x}) + \theta_i\| + m|\underline{x}| M(\underline{y}_k) \\ &\leq nY_k \max_{1 \leq i \leq n} \|L_i(\underline{x}) + \theta_i\| + m|\underline{x}| Y_{k+1}^{-w} \\ &\leq nY_k \max_{1 \leq i \leq n} \|L_i(\underline{x}) + \theta_i\| + \frac{Y_k^{-\delta}}{2}, \end{aligned}$$

from which follows the lower bound

$$\begin{aligned} \|A\underline{x} + \underline{\theta}\| &= \max_{1 \leq i \leq n} \|L_i(\underline{x}) + \theta_i\| \geq \frac{1}{2n} Y_k^{-1-\delta} \\ &\geq (2m)^{-(\delta+1)/(w-\delta)} (2n)^{-1} |\underline{x}|^{-(\delta+1)/(w-\delta)}. \end{aligned}$$

We deduce that

$$w(A, \underline{\theta}) \leq \frac{\delta + 1}{w - \delta}.$$

Choosing  $\delta$  and  $w$  arbitrarily close to 0 and to  $\hat{w}({}^tA)$  respectively, we obtain the first upper bound of (10).

In order to prove the second upper bound of (10), we take again the preceding argumentation using now the estimation (4) instead of (3). Let us fix two real numbers  $\delta$  and  $w$  satisfying

$$0 < \delta < w < w({}^tA).$$

Let  $k$  be an integer  $\geq 1$  such that  $M_k \leq Y_k^{-w}$ . Since  $w < w({}^tA)$ , there exist infinitely many such integers  $k$ . Let  $\underline{x}$  be an integer  $m$ -tuple with norm  $|\underline{x}| \leq X_k := Y_k^{w-\delta}/(2m)$ . Combining (4), (11) and (12), we obtain

$$\begin{aligned} Y_k^{-\delta} &\leq n |\underline{y}_k| \max_{1 \leq i \leq n} \|L_i(\underline{x}) + \theta_i\| + m |\underline{x}| M(\underline{y}_k) \\ &\leq n Y_k \max_{1 \leq i \leq n} \|L_i(\underline{x}) + \theta_i\| + m X_k Y_k^{-w}, \end{aligned}$$

from which we deduce that

$$\max_{1 \leq i \leq n} \|L_i(\underline{x}) + \theta_i\| \geq \frac{1}{2n} Y_k^{-1-\delta} = (2m)^{-(\delta+1)/(w-\delta)} (2n)^{-1} X_k^{-(\delta+1)/(w-\delta)}.$$

Recall now that the above lower bound holds for any integer point with norm  $\leq X_k$  and for infinitely many integers  $k \geq 1$ . Noting that the sequence  $(X_k)_{i \geq 1}$  tends to infinity, it follows that

$$\hat{w}(A, \underline{\theta}) \leq \frac{\delta + 1}{w - \delta}.$$

Choosing  $\delta$  and  $w$  arbitrarily close to 0 and to  $w({}^tA)$  respectively, we obtain the second upper bound of (10).

Furthermore, the preceding arguments enable us to establish Proposition 1. The first assertion follows immediately from the Theorem, since we just have to pick an  $n$ -tuple  $\underline{\theta}$  out of a set of full Lebesgue measure. The proof of the second assertion needs more work. We begin by extracting some subsequence from the sequence of best approximations  $(\underline{y}_i)_{i \geq 1}$ .

We claim that there exists an increasing function  $\varphi : \mathbf{Z}_{\geq 1} \rightarrow \mathbf{Z}_{\geq 1}$  satisfying  $\varphi(1) = 1$  and, for any integer  $i \geq 2$ ,

$$(13) \quad Y_{\varphi(i)} \geq (9n)^{1/2} Y_{\varphi(i-1)} \quad \text{and} \quad Y_{\varphi(i-1)+1} \geq (9n)^{-1} Y_{\varphi(i)}.$$

The function  $\varphi$  is constructed in the following way. Let  $j > j'$  be two indices such that  $Y_j \geq (9n)^{1/2} Y_{j-1}$  and  $Y_{j'} \geq (9n)^{1/2} Y_{j'-1}$ . Suppose that  $j' - 1 = \varphi(h')$ , and that the function  $\varphi$  has already been defined for  $1 \leq i \leq h'$ . We set  $j - 1 = \varphi(h)$  for some  $h > h'$ , which will be specified later. We let  $\varphi(h - 1)$  be the largest index  $t \geq j'$  for which  $Y_{\varphi(h)} \geq (9n)^{1/2} Y_t$ . We let  $\varphi(h - 2)$  be the largest index  $t \geq j'$  for which  $Y_{\varphi(h-1)} \geq (9n)^{1/2} Y_t$ , and so on until it does not exist any index  $t$  as above. We have just defined  $\varphi(h), \varphi(h - 1), \dots, \varphi(h - h_0)$ . Then, we set  $h = h_0 + h' + 1$ , and we check that the inequalities (13) are satisfied for  $i = h' + 1, \dots, h_0 + h' + 1$ .

This process does not apply when there are only finitely many indices  $j$  such that  $Y_j \geq (9n)^{1/2} Y_{j-1}$ . In this case, we denote by  $g$  the largest of these indices ( $g = 1$  if there is none) and we apply the above process to construct the initial values of the function  $\varphi$  up

to  $g = \varphi(h)$ . Next, we define  $\varphi(h+1)$  as the smallest index  $t$  for which  $Y_t \geq (9n)^{1/2} Y_{\varphi(h)}$ . Then, we observe that  $Y_{\varphi(h+1)-1} < (9n)^{1/2} Y_{\varphi(h)}$  and

$$Y_{\varphi(h)+1} \geq Y_{\varphi(h)} > (9n)^{-1/2} Y_{\varphi(h+1)-1} > (9n)^{-1} Y_{\varphi(h+1)},$$

as required. We continue in this way, defining  $\varphi(h+2)$  as the smallest index  $t$  for which  $Y_t \geq (9n)^{1/2} Y_{\varphi(h+1)}$ , and so on. The inequalities (13) are then satisfied.

The first inequalities in (13) enable us to satisfy the assumptions of Lemma 2, page 86, from [6] for the sequence of integer vectors  $(\underline{y}_{\varphi(i)})_{i \geq 1}$  with  $k = 3$ . Consequently, there exists a real  $n$ -tuple  $\underline{\theta}$  such that

$$(14) \quad \|y_{\varphi(i),1}\theta_1 + \dots + y_{\varphi(i),n}\theta_n\| \geq \frac{1}{4}, \quad \text{for all } i \geq 1.$$

Let  $\underline{x}$  be a non-zero integer  $m$ -tuple and let  $k$  be the index defined by the inequalities

$$Y_{\varphi(k)} \leq 9n(8m)^{m/n} |\underline{x}|^{m/n} < Y_{\varphi(k+1)}.$$

Taking into account (3), (5), (14) and (11) applied with  $\underline{y} = \underline{y}_{\varphi(k)}$ , we have

$$\frac{1}{4} \leq (9n^2)(8m)^{m/n} |\underline{x}|^{m/n} \|A\underline{x} + \underline{\theta}\| + m|\underline{x}| Y_{\varphi(k)+1}^{-n/m}.$$

By construction of the subsequence  $(Y_{\varphi(i)})_{i \geq 1}$ , we have  $Y_{\varphi(k)+1}^{-1} \cdot Y_{\varphi(k+1)} \leq 9n$ , so that

$$\frac{1}{4} \leq (9n^2)(8m)^{m/n} |\underline{x}|^{m/n} \|A\underline{x} + \underline{\theta}\| + m(8m(9n)^{n/m})^{-1} (9n)^{n/m},$$

and

$$\|A\underline{x} + \underline{\theta}\| \geq \frac{1}{72n^2(8m)^{m/n}} |\underline{x}|^{-m/n},$$

as announced.  $\square$

## 5. The Corollary.

Our Theorem reduces the determination of the measure of generic density

$$\inf_{\underline{\theta} \in \mathbb{R}^n} w(A, \underline{\theta}) = \frac{1}{\hat{w}({}^t A)}$$

of the group  $\Gamma$  to the computation of the exponent  $\hat{w}({}^t A)$ . Any upper bound of  $\hat{w}({}^t A)$  implies a uniform lower bound for the exponents of approximation  $w(A, \underline{\theta})$ . When

$$A = (\xi, \dots, \xi^n),$$

for a real transcendental number  $\xi$ , there are known upper bounds for  $\hat{w}(A)$  and  $\hat{w}(^tA)$ , which depend only upon  $n$ . Coming back to the specific notations of [5]:

$$\hat{w}(A) = \hat{w}_n(\xi) \quad \text{and} \quad \hat{w}(^tA) = \hat{\lambda}_n(\xi),$$

we have

$$\hat{\lambda}_n(\xi) \leq \frac{1}{\lceil n/2 \rceil} \quad \text{and} \quad \hat{w}_n(\xi) \leq 2n - 1.$$

The first upper bound is the main result of [18], while Theorem 2b of [8] is equivalent to the second one. Actually, the results of [18] and [8] are slightly sharper: the refinement stated below Proposition 1 hold in these cases. Combined with Lemma 3, they yield the assertions (i) and (ii) of the Corollary.

In degree  $n = 2$ , the exact upper bounds for the functions  $\hat{\lambda}_2(\xi)$  and  $\hat{w}_2(\xi)$  are known: Roy [19] and Arbour & Roy [1] have proved that

$$\hat{\lambda}_2(\xi) \leq \frac{\sqrt{5} - 1}{2} \quad \text{and} \quad \hat{w}_2(\xi) \leq \frac{\sqrt{5} + 3}{2},$$

and both equalities hold when  $\xi$  is a Fibonacci continued fraction. The assertion (iii) of the Corollary is then the translation of our Theorem in this particular case.

**Remarks.** (i) The exponents  $\hat{\lambda}_2(\xi)$  and  $\hat{w}_2(\xi)$  have been computed more generally in [5] for any Sturmian continued fraction  $\xi$  of irrational angle  $\varphi$ . It turns out that

$$\hat{\lambda}_2(\xi) > \frac{1}{2} \quad \text{and} \quad \hat{w}_2(\xi) > 2$$

when the partial quotients in the continued fraction expansion of  $\varphi$  are bounded. The associated subgroups  $\Gamma$

$$\mathbf{Z} + \mathbf{Z}\xi + \mathbf{Z}\xi^2 \quad \text{and} \quad \mathbf{Z} \begin{pmatrix} \xi \\ \xi^2 \end{pmatrix} + \mathbf{Z}^2$$

are then dense in  $\mathbf{R}$  and  $\mathbf{R}^2$  respectively, and their generic exponents of density are less than 2 and 1/2, respectively.

(ii) When  $\xi$  is some real number connected with the Fibonacci continued fraction, Roy [20] has proved that we have a lower bound of the form

$$|P(\xi) + \xi^3| \gg H(P)^{-(1+\sqrt{5})/2}$$

for any quadratic polynomial  $P(X)$  with integer coefficients. This means that the number  $\theta = \xi^3$  shares the almost sure property stated in the part (iii) of the Corollary. The difficult point in Roy's proof is to verify that a lower bound similar to (14) is valid for  $\underline{\theta} = (\xi^3)$ .

## 6. Hausdorff dimension.

The Hausdorff dimension is a useful tool to discriminate between sets of Lebesgue measure zero and thus to prove the existence of real numbers having fine properties of Diophantine approximation (see, for instance, Chapter 5 of [3]). We denote it by  $\dim$  and direct the reader to the books of Falconer [10, 11] for the definition and the properties of Hausdorff dimension.

Let  $A$  be a matrix in  $M_{n,m}(\mathbf{R})$  and let  $w > 1/\hat{w}({}^tA)$  be a real number. In view of our Theorem, it is natural to ask whether there exists a real  $n$ -tuple  $\underline{\theta}$  such that  $w(A, \underline{\theta}) = w$ . As a first step, we wish to compute the Hausdorff dimension of the null set

$$\mathcal{U}_w(A) = \left\{ \underline{\theta} \in \mathbf{R}^n : \|A\underline{x} + \underline{\theta}\| \leq \frac{1}{|\underline{x}|^w} \text{ for infinitely many } \underline{x} \text{ in } \mathbf{Z}^m \right\}.$$

This question has been solved in the simplest case when  $A = (\xi)$ , for any given irrational number  $\xi$ , by Bugeaud [2] and, independently, by Schmeling & Troubetzkoy [21]. We then have  $w((\xi), \underline{\theta}) = 1$  for almost all real numbers  $\theta$  and the Hausdorff dimension of the set  $\mathcal{U}_w((\xi))$  is equal to  $1/w$ , for any  $w \geq 1$ .

This question remains unanswered for other matrices  $A$ . Nevertheless, Bugeaud & Chevallier [4] have proved that, for almost all matrices  $A$  in  $M_{n,m}(\mathbf{R})$  and for any real number  $w \geq m/n$ , we have

$$\dim \mathcal{U}_w(A) = \dim \{ \underline{\theta} \in \mathbf{R}^n : w(A, \underline{\theta}) \geq w \} = \frac{m}{w}.$$

Besides, Theorem 3 of [4] asserts that, if  $A$  is a column matrix, then

$$\dim \mathcal{U}_w(A) = \dim \{ \underline{\theta} \in \mathbf{R}^n : w(A, \underline{\theta}) \geq w \} = \frac{1}{w}$$

for any real number  $w \geq 1$ . In the above examples, the sets  $\{ \underline{\theta} \in \mathbf{R}^n : w(A, \underline{\theta}) \geq w \}$  and  $\mathcal{U}_w(A)$  have the same Hausdorff dimension (although the first one contains the second).

The results of [4] indicate that the situation is much more complicated when the matrix  $A$  is not of the form  $(\xi)$ . It seems to us that the determination in our general framework of the Hausdorff dimension of the sets  $\mathcal{U}_w(A)$  is a quite difficult problem. Nonetheless, it is possible to show that this dimension is strictly less than  $n$ , whenever  $w > 1/\hat{w}({}^tA)$ .

**Proposition 2.** *Let  $A$  be a matrix in  $M_{n,m}(\mathbf{R})$  and let  $w$  be a real number  $> 1/\hat{w}({}^tA)$ . The set*

$$\{ \underline{\theta} \in \mathbf{R}^n : w(A, \underline{\theta}) \geq w \}$$

*has Lebesgue measure zero, and its Hausdorff dimension is strictly less than  $n$ .*

*Proof.* Set

$$\delta = \frac{w\hat{w}({}^tA) - 1}{2 + w + 1/\hat{w}({}^tA)}$$

and notice that the inequality (6) determines in the hypercube  $[0, 1]^n$  a subset contained in the union of at most  $c(n) Y_i \cdot Y_i^{(\delta+1)(n-1)}$  hypercubes with edge  $Y_i^{-\delta-1}$ , where  $c(n)$  denotes some suitable constant, depending only upon  $n$ . Since, by Lemma 1, the series

$$\sum_{i \geq 1} Y_i^{1+(\delta+1)(n-1)} \cdot Y_i^{-(\delta+1)s}$$

converges for any  $s > n - 1 + 1/(\delta + 1)$ , the Hausdorff–Cantelli Lemma (cf. for example [3], Chapter 5) ensures us that the Hausdorff dimension of the set

$$\mathcal{V}_\delta = \{\underline{\theta} \in \mathbf{R}^n : \|y_{i,1}\theta_1 + \cdots + y_{i,n}\theta_n\| < Y_i^{-\delta} \text{ for infinitely many } i\}$$

is bounded from above by  $n - 1 + 1/(\delta + 1)$ . This is strictly less than  $n$  since  $\delta$  is positive. Let  $\underline{\theta}$  be in the complement of  $\mathcal{V}_\delta$ , and follow again the proof of our Theorem. The inequality (12) is then satisfied for any sufficiently large integer  $i$ . Thus, we have the upper bound

$$w(A, \underline{\theta}) \leq \frac{\delta + 1}{\hat{w}({}^t A) - \delta},$$

and, by our choice of  $\delta$ ,

$$w(A, \underline{\theta}) \leq \frac{1}{2} \left( \frac{1}{\hat{w}({}^t A)} + w \right) = w - \frac{1}{2} \left( w - \frac{1}{\hat{w}({}^t A)} \right).$$

Consequently, the set  $\mathcal{U}_w(A)$  is contained in  $\mathcal{V}_\delta$ . This remark concludes the proof of Proposition 2.  $\square$

In view of the results of [19, 20, 5], it may be possible to determine the Hausdorff dimensions of the sets  $\mathcal{U}_w(A)$  and  $\mathcal{U}_w({}^t A)$ , when  $A = (\xi, \xi^2)$  and  $\xi$  is a Sturmian continued fraction. We plan to return to these questions later. Let us simply remark that our Proposition 2 implies that  $\dim \mathcal{U}_2(A) < 1$  and  $\dim \mathcal{U}_{1/2}({}^t A) < 2$ , unlike in the generic situation.

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